JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 6, 387-393 (1963)

Some Results for the Solutions of a Certain System of Differential Equations

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1. We shall consider the system of differential equations

$$\frac{dx}{dt} = f(x; t) \tag{1.1}$$

where x and f are real n-dimensional column vectors with components $x_1, x_2, \cdots, x_n$ and $f_1, f_2, \cdots, f_n$ respectively. For the function f we shall assume that the partial derivatives $(\partial f_i/\partial x_j)$ $(1 \leqslant i, j \leqslant n)$ exist and that f, $(\partial f_i/\partial x_j)$ $(1 \leqslant i, j \leqslant n)$ are continuous for all x, t considered. Throughout what follows the norm $\| \xi \|$ of any given (real) n dimensional vector ξ is taken to be the Euclidean norm $(\xi, \xi)^{1/2}$, where (ξ, ξ) is a scalar product. The first main result of the paper is the following estimate for the solutions of (1.1).

THEOREM 1. *Suppose that there exists a symmetric, positive definite $n \times n$ matrix $A = (\alpha_{ij})$ with α_{ij} $(1 \leqslant i, j \leqslant n)$ all real constants such that, setting*

$$d_{ij} = \sum_{k=1}^{n} \alpha_{ik} \frac{\partial f_k}{\partial x_j},$$

each characteristic root λ_k $(k = 1, 2, \cdots, n)$ of the symmetrized generalized Jacobi matrix $(\frac{1}{2}(d_{ij} + d_{ji}))$ satisfies

$$\lambda_k \leq -\delta < 0 \tag{1.2}$$

uniformly in x and $t \geqslant t_0$. Then every solution $x(t)$ of (1.1) satisfies[1]

$$\| x(t) \| \leqslant \left\{ e^{-1/2(p\delta/\alpha)t} \left[C_1 + C_2 \int_{t_0}^{t} \| f(0; \tau) \|^p e^{1/2(p\delta/\alpha)\tau} \, d\tau \right] \right\}^{1/p} \tag{1.3}$$

for all $t \geqslant t_0$, where $C_1 = C_1(\delta, t_0, A, x(t_0)) > 0$ and $C_2 = C_2(\delta, A) > 0$ are

[1] The constant $\frac{1}{2}(p\delta/\alpha)$ in each of the exponential terms in (1.3) can be improved to $(p\mu/\alpha)$, where μ is any constant such that $0 \leqslant \mu < \delta$.

constants depending only on the arguments explicitly displayed; α is the largest characteristic root of A; and p is any constant such that $1 \leqslant p \leqslant 2$.

Observe that if

$$p = 1 \qquad \text{and} \qquad \alpha_{ij} \equiv \delta_{ij},$$

where δ_{ij} is the Kronecker delta, the corresponding result (1.3) is comparable with Rosen's estimate [1; Theorem 5] for solutions of (1.1).

In the special case when f satisfies one or other of the conditions:

$$\max_{t \geqslant t_0} \| f(0; t) \| < \infty, \qquad \int_{t_0}^{\infty} \| f(0; t) \|^p \, dt < \infty,$$

the present theorem leads to the result that, subject to the conditions on A and f, every solution $x(t)$ of (1.1) satisfies

$$\| x(t) \| \leq C \tag{1.4}$$

as $t \to \infty$, where the constant C, $0 < C < \infty$, depends only on A and f. This boundedness result (1.4) has already been obtained by Demidovič (2; Theorem 1] under a weaker (1.2), but Demidovič's result is stated and proved only for the case when $\| f(0; t) \|$ is finitely bounded for $t \geqslant t_0$.

Observe also that when f is such that $f(0; t) \equiv 0$, Theorem 1 implies, among other things, that every solution $x(t)$ of (1.1) satisfies

$$x(t) \to 0 \qquad \text{as} \qquad t \to \infty \tag{1.5}$$

provided that each λ_k satisfies (1.2). The result (1.5) is, however, not confined exclusively to Eq. (1.1) for which $f(0; t) \equiv 0$, as is shown by our next result.

THEOREM 2. *If, given any real number b $(0 \leqslant b < \infty)$, $f(x, t)$ satisfies*

$$\| f(x; t) \| \leqslant \varphi(b) \qquad \textit{for} \qquad \| x \| \leqslant b, \tag{1.6}$$

uniformly in $t \geqslant t_0$, where $\varphi(b)$ is a continuous function depending only on b, and if further

$$\int_{t_0}^{\infty} \| f(0; t) \|^p \, dt < \infty \tag{1.7}$$

where p is any constant in the range $1 \leqslant p \leqslant 2$, then, subject to the conditions on A and f in Theorem 1, every solution $x(t)$ of (1.1) satisfies (1.5).

An obvious example of (1.1) for which the condition (1.6) is satisfied is the case when f is of the form:

$$f = F(x) + e(t)$$

where F, e are continuous vector functions depending only on the arguments displayed, and $\| e(t) \|$ is finitely bounded for all sufficiently large t.

2. *A lemma.* For the proof of Theorem 1, we shall require the following preliminary result:

LEMMA. *Let $g(x; t)$ be a real n-dimensional column vector with components $g_1, g_2, \cdots, g_n$ and suppose that g is continuous, and that its partial derivatives $(\partial g_i/\partial x_j)$ $(1 \leqslant i, j \leqslant n)$ exist and are continuous, for all x, t considered. If there is a constant N, $-\infty < N < \infty$, such that the characteristic roots ν_k $(k = 1, 2, \cdots n)$ of the matrix*

$$\frac{1}{2}\left(\frac{\partial g_i}{\partial x_j} + \frac{\partial g_j}{\partial x_i}\right)$$

satisfy

$$\nu_k \leq N \qquad (k = 1, 2, \cdots, n) \tag{2.1}$$

uniformly in x and $t \geqslant t_0$, then for any two given vectors x, h whatever the scalar product $(g(x+h; t) - g(x; t), h)$ satisfies

$$(g(x+h; t) - g(x; t), h) \leqslant N \| h \|^2 \tag{2.2}$$

for all $t \geqslant t_0$.

PROOF. The result (2.2) is essentially the same as the right hand inequality in the conclusion of Demidovič's basic lemma [2; Section 2], but the following proof of the result appears more direct than that given in [2; Section 2].

By the mean values theorem

$$g(x+h; t) - g(x; t) = \Gamma h,$$

Γ being the $n \times n$ matrix (γ_{ij}) where

$$\gamma_{ij} = \frac{\partial g_i}{\partial x_j}(x + \theta_i h; t) \tag{2.3}$$

and $\theta_i = \theta_i(x; t)$ satisfies $0 < \theta_i < 1$. Hence

$$\begin{aligned}(g(x+h; t) - g(x; t), h) &= (\Gamma h, h)\\ &= (\Gamma^* h, h)\end{aligned}$$

for every x, h, t, where Γ^* is the $n \times n$ matrix $(\frac{1}{2}(\gamma_{ij} + \gamma_{ji}))$ and γ_{ij} is given by (2.3). But, if (2.1) holds then, since Γ^* is symmetric,

$$(\Gamma^* h, h) \leq N(h, h), \qquad t \geqslant t_0$$

and so we have the lemma.

3. **Proof of Theorem** 1. Let $x = x(t)$ be any solution of (1.1) and set

$$V(t) = (Ax(t), x(t))$$

where A is the matrix defined in the theorem. Since A is symmetric and positive definite it is clear that

$$\alpha \| x(t) \|^2 \geqslant V(t) \geqslant \alpha' \| x(t) \|^2 \tag{3.1}$$

for all t, where $\alpha > 0$ and $\alpha' > 0$ are the greatest and the least characteristic roots respectively of A. We shall now show that, under the conditions of Theorem 1, $V(t)$ also satisfies

$$\frac{dV}{dt} \leq -2\delta_0 V + C_3 \|f(0; t)\| V^{1/2}, \tag{3.2}$$

for all $t \geqslant t_0$, where $\delta_0 = \delta/\alpha$ and C_3, $0 < C_3 < \infty$, is a constant depending only on A. In fact since A is symmetric it is evident that

$$\begin{aligned} \tfrac{1}{2}\frac{dV}{dt} &= (Af(x; t), x) \\ &= (Af(x; t) - Af(0; t), x) + (Af(0; t), x) \\ &\equiv \qquad U_1 \qquad\qquad + \qquad U_2, \end{aligned} \tag{3.3}$$

say. If (1.2) holds then, by the lemma,

$$\begin{aligned} U_1 &\leqslant -\delta \| x \|^2 \qquad (t \geqslant t_0) \\ &\leqslant -(\delta/\alpha) V \end{aligned}$$

by (3.1). Concerning U_2 an immediate application of Cauchy's inequality gives that

$$\begin{aligned} | U_2 | &\leqslant \left\{ \sum_{1 \leqslant i,j \leqslant n} \alpha_{ij}^2 \right\}^{1/2} \|f(0; t)\| \, \| x \| \\ &\leqslant \left\{ \sum_{1 \leqslant i,j \leqslant n} \alpha_{ij}^2 \right\}^{1/2} \|f(0; t)\| (V/\alpha')^{1/2} \end{aligned}$$

by (3.1), and (3.2) now follows on substituting these estimates of U_1, U_2 in (3.3).

For further progress in the proof of Theorem 1, it is convenient to rewrite (3.2) in-the form:

$$\frac{dV}{dt} + \delta_0 V \leqslant -\delta_0 V + C_3 \|f(0; t)\| V^{1/2}. \tag{3.4}$$

Now let p be any constant in the range $1 \leqslant p \leqslant 2$ and set

$$\beta = 1 - \tfrac{1}{2}p. \tag{3.5}$$

Because $1 \leqslant p \leqslant 2$, the constant β evidently lies in the range $0 \leqslant \beta \leqslant \frac{1}{2}$ and thus (3.4) may, in turn, be reset in the following form:

$$\frac{dV}{dt} + \delta_0 V \leq V^{\beta} U \tag{3.6}$$

where

$$U \equiv V^{(1/2-\beta)} \{C_3 \| f(0; t) \| - \delta_0 V^{1/2}\}. \tag{3.7}$$

From the definitions of β and U it is readily verified that

$$U \leqslant C_4 \| f(0; t) \|^p \tag{3.8}$$

for all t, where $C_4 = C_4(C_3, \delta_0) > 0$ is a constant. Indeed if f and V are such that

$$C_3 \| f(0; t) \| \leqslant \delta_0 V^{1/2}$$

then (3.7) gives that

$$U \leqslant 0$$

which is certainly included in (3.8); and if, on the other hand,

$$C_3 \| f(0; t) \| > \delta_0 V^{1/2}$$

then, by (3.7),

$$\begin{aligned} U &\leqslant V^{(1/2-\beta)} C_3 \| f(0; t) \| \\ &< [\{C_3 \| f(0; t) \| / \delta_0\}^2]^{(1/2-\beta)} C_3 \| f(0; t) \| \\ &= C_3^p \delta_0^{(2\beta-1)} \| f(0; t) \|^p \end{aligned}$$

by (3.5) and this is (3.8) with $C_4 = C_3^p \delta_0^{(2\beta-1)}$. Hence (3.8) holds for all t and, on substituting this in (3.6), we have that

$$\frac{dV}{dt} + \delta_0 V \leqslant C_4 V^{\beta} \| f(0; t) \|^p \tag{3.9}$$

for all $t \geqslant t_0$. Multiply both sides of this inequality by $e^{\zeta t}$, where

$$\zeta = (1 - \beta)\, \delta_0,$$

and the result of the multiplication can be shown equivalent to:

$$\frac{d}{dt} \{V^{(1-\beta)} e^{\zeta t}\} \leqslant (1 - \beta)\, C_4 \| f(0; t) \|^p e^{\zeta t}.$$

Integrating both sides of this from t_0 to T, $T \geqslant t_0$, we obtain, after due simplifications,

$$\{V(T)\}^{(1-\beta)} \leqslant e^{-\zeta T}\left\{e^{\zeta t_0}[V(t_0)]^{(1-\beta)} + (1-\beta)\, C_4 \int_{t_0}^{T} \|f(0;t)\|^p e^{\zeta t}\, dt\right\}.$$

Insert the values

$$1-\beta = \tfrac{1}{2}p, \qquad \zeta = \tfrac{1}{2}p\,\frac{\delta}{\alpha}$$

in this result, and the conclusion of the theorem then follows on making use of (3.1).

4. Proof of Theorem 2. Let $V(t)$ be the function defined at the beginning of Section 3. To prove Theorem 2, it will be sufficient to show that, under the stated conditions on A and f,

$$\frac{d}{dt}V(t) = 0(1) \qquad \text{and} \qquad \int_{t_0}^{t} V(\tau)\, d\tau = 0(1) \tag{4.1}$$

as $t \to \infty$; for then, since $V(t) \geqslant 0$ for all t, one can show by a routine analysis[2] that

$$V(t) \to 0 \qquad \text{as} \qquad t \to \infty$$

and this, by (3.1) implies (1.5).

To deal with the first part of (4.1) note that if (1.7) holds then by Theorem 1 there is a constant C_5, $0 < C_5 < \infty$, depending on t_0, $x(t_0)$, A and f such that

$$\| x(t) \| \leq C_5, \qquad t \geqslant t_0. \tag{4.2}$$

Next note from the first line of (3.3) that

$$\tfrac{1}{2}\left|\frac{dV}{dt}\right| \leqslant \left\{\sum_{1\leqslant i,j\leqslant n} \alpha_{ij}^2\right\}^{1/2} f(x;t) \|\, \| x \|$$

for all x, t. Clearly if (1.6) holds, then (4.2) implies that

$$\left|\frac{dV}{dt}\right| \leqslant 2C_5 \left\{\sum_{1\leqslant i,j\leqslant n} \alpha_{ij}^2\right\}^{1/2} \varphi(C_5) < \infty, \qquad t \geqslant t_0$$

and thus we have the first part of (4.1).

To prove the second part we start with the inequality (3.9). Integrating this from t_0 to t ($t \geqslant t_0$) we find, since $V \geqslant 0$, that

$$\delta_0 \int_{t_0}^{t} V(\tau)\, d\tau \leqslant V(t_0) + C_4 \int_{t_0}^{t} V^{\beta}(\tau) \|f(0;\tau)\|^p\, d\tau$$

$$= V(t_0) + C_4 V^{\beta}(t') \int_{t_0}^{t} \|f(0;\tau)\|^p\, d\tau, \qquad t_0 < t' < t, \tag{4.3}$$

[2] Such as is used, for example, by Lefschetz in (3; p. 273).

by the mean value theorem for integrals. If (4.2) holds then, by (3.1),

$$V^{\beta}(t') \leqslant (\alpha C_5^2)^{\beta}$$

for all $t' \geqslant t_0$; and thus, provided that (1.7) holds, (4.3) gives that

$$\int_{t_0}^{t} V(\tau)\, d\tau = 0(1)$$

as $t \to \infty$. This completes the proof of (4.1) and Theorem 2 now follows as stated.

References

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